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A Dynamical Phase Transition in an Infinite Particle System

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The local equilibrium picture of the time evolution of a gas may have to be modified in the presence of shocks in order to admit statistical mixtures of pure states in the hydrodynamic description. An example drawn from a stochastic many-particle model (asymmetric zero-range model) is described.

KEY WORDS: Hydrodynamical behavior; phase transitions; shock waves; zero-range process.

1. INTRODUCTION

Implicit in the hydrodynamic picture of the time evolution of a gas or fluid is the assumption of local equilibrium. On a microscopic (space and time) scale the fluid is in equilibrium; on a macroscopic scale, the equilibrium parameters evolve according to hydrodynamic equations. Thus a scaling limit—the limit of vanishing gradients—is implicit in this description from the outset. The overall picture should be valid for a suitable class of "local equilibrium" states which can then be said to exhibit "hydrodynamic behavior."

A rigorous proof of the validity of this picture for any realistic model of a fluid may be beyond our capabilities at present. But as a recent survey makes clear,[(1)] for some simplified models—"caricatures" of a real gas—the local equilibrium picture can indeed be justified. These include models with Newtonian dynamics (ideal gas, hard rods in one dimension) and models with stochastic dynamics (simple exclusion process, zero-range process,...). Experience with these models lead the authors in Ref. 1 to propose a hierarchy of ergodic properties which a system may possess, driving it to

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local equilibrium; thus, local mixing, local ergodicity, etc. This paper concerns a particular clear example, intended to elucidate the distinctions between these definitions, and to relate the breakdown of local ergodicity to the formation of a shock wave. Before discussing the model we review the relevant material from Ref. 1. For simplicity we omit some of the technical assumptions.

Let our infinite particle system have a family $\mathscr{E}$ of translation-invariant, extremal equilibrium measures, labeled by the values of a single (locally conserved) quantity, e.g., the particle density ρ. (Of course, the equilibrium states of a Newtonian fluid are labeled by the values of the five locally conserved fields, namely, the mass, energy, and the three components of the momentum density.) We write $\mathscr{E} = \{v_\rho\}_{0 \leqslant \rho < \infty}$. Let T_t^* denote time evolution of states (probability measures on phase space) under the dynamics, and let $\bar{d}$ be a bounded metric for the weak topology on states. A state μ is called *locally mixing* if

$$\lim_{t \to \infty} \sup_x \bar{d}(D_x T_t^* \mu, \mathscr{E}) = 0 \tag{1.1}$$

where D_x is spatial translation by x. μ is called *weakly locally ergodic* if for every $\delta > 0$ there is a time T_δ, $T_\delta \to \infty$ as $\delta \to 0$, and $t_\delta < \infty$ such that

$$\sup_{t \geqslant t_\delta} \sup_x P^\mu \left\{ \eta\colon \bar{d}\left(D_x \frac{1}{T_\delta} \int_t^{t+T_\delta} ds\, \delta_{\eta_s}, \mathscr{E} \right) > \delta \right\} < \delta \tag{1.2}$$

In (1.2) η_t stands for the configuration of the system at time t, and δ_η for the point mass at η. P^μ is the path measure, with initial measure μ. Finally, μ is called *locally ergodic* if for every $\delta > 0$

$$\sup_{t \geqslant t_\delta} \sup_x \bar{d}\left(D_x \frac{1}{T_\delta} \int_t^{t+T_\delta} ds\, T_s^* \mu, \mathscr{E} \right) < \delta \tag{1.3}$$

These three definitions represent increasingly strong forms of local ergodicity. It is the distinction between weak local ergodicity and local ergodicity (or mixing) which we shall illustrate in this paper; from the definitions, it is clear that the difference lies in the dependence of the local equilibrium parameters on the path of the process. That local mixing implies local ergodicity implies weak local ergodicity follows from Birkoff's theorem and a compactness argument; see Ref. 1.

If a state is locally mixing or ergodic, an "equilibrium profile" $\rho(x, t)$ is at least approximately defined for large t. To obtain sharper results—permitting one to investigate the hydrodynamic equation of the system—one must assume that there is a fixed "hydrodynamic scaling" for the system, and select a suitable family of states defining an equilibrium profile at time zero. Let μ^ε, $\varepsilon > 0$ be a family of states such that there exists a function $\rho(\xi, 0)$ with

$$\lim_{\substack{\varepsilon \to 0 \\ \varepsilon x \to \xi}} \bar{d}(D_x \mu^\varepsilon, \nu_{\rho(\xi,0)}) = 0 \tag{1.4}$$

for all ξ, and let $t(\varepsilon, \tau)$ be a function increasing in τ and decreasing in ε. Following Ref. 1 we call $\{\mu^\varepsilon\}_{\varepsilon > 0}$ a *local equilibrium distribution* if there is a function $\rho(\xi, t)$ such that

$$\lim_{\substack{\varepsilon \to 0 \\ \varepsilon x \to \xi}} \bar{d}(D_x T^*_{t(\varepsilon,\tau)} \mu^\varepsilon, \nu_{\rho(\xi,\tau)}) = 0 \tag{1.5}$$

One expects to find a local equilibrium distribution with any given initial profile if a large class of locally ergodic (or mixing) states exists.

The model we discuss is called the asymmetric zero-range process (AZRP), and is one of a class of models introduced by Spitzer in Ref. 2. In a tour de force of coupling arguments, the existence of local equilibrium distributions whose equilibrium profile $\rho(\xi, \tau)$ satisfied a differential equation—the hydrodynamic equation of the system—was proven by Andjel and Kipnis in Ref. 3. (See also Section 4 of the present paper.) The hydrodynamic equation turns out to have the form of a (nonlinear) "differential conservation law," in the terminology of Ref. 4, and consequently admits shock wave solutions. We shall show that these solutions are associated—microscopically, in our model—with the breakdown of local ergodicity.

2. THE ASYMMETRIC ZERO-RANGE PROCESS

This is a Markov process with state space $X = (\mathbb{Z}^+)^{\mathbb{Z}}$. $\eta \in X$ represents a configuration of particles located at sites $x \in \mathbb{Z}$; $\eta(x)$ is the occupation number at site x. The dynamics is given by the following rule. At each site there is a clock which rings after a waiting time with an exponential distribution (Poisson process); when a clock rings, if that site is occupied one particle jumps to the next site on the right. The clocks are independent of each other and of the occupation variables. Since the rate of jumping of each particle depends only on the occupation variable of the site it occupies, the interaction is said to be zero range; since particles jump only to the right, the process is said to be asymmetric; hence the name.

Usually one does not construct the variables directly as described; instead, one writes down an infinitesimal generator and constructs the process using the Hille–Yoshida theorem or by proving existence and uniqueness of the martingale problem (see Ref. 5 and references therein). For our process the generator L acting on cylinder functions f is given by

$$[Lf](\eta)=\sum_{x\in\mathbb{Z}} 1_{[\eta(x)>0]}(f(\eta^{x,x+1})-f(\eta)) \tag{2.1}$$

In (2.1) 1_A is the characteristic function of the event A and

$$\eta^{x,x+1}(y)=\begin{array}{ll}\eta(x)-1, & y=x\\ \eta(x+1)+1, & y=x+1\\ \eta(y), & y\neq x,\ x+1\end{array} \tag{2.2}$$

The Markov process associated to L is then determined (formally) by the formula

$$E^{\eta}f(\eta_t)=[e^{Lt}f](\eta) \tag{2.3}$$

where η_t is the configuration of the process at time t and E^{η} is expectation with respect to the process with initial configuration η.

Some additional notation: Ω will denote the path space of functions: $t\to\eta_t\in X$ which are right continuous with left limits (X is given the product topology). Given a probability measure μ on X, P^{μ} will denote the corresponding path measure with μ as initial state. If $A\subset\mathbb{Z}$, $0<|A|<\infty$, $\eta(A)$ denotes the function: $\eta\to\prod_{x\in A}\eta(x)$.

Let $\{\nu_\rho\}_{0\leqslant\rho<\infty}$ be the family of translation-invariant, extremal, equilibrium states of the AZRP,[6] indexed by mean density: $\nu_\rho(\eta(x))=\rho$ [ν_ρ is a product measure for which $\eta(x)$ has a geometric distribution]. Given a function ρ_0: $\mathbb{R}\to\mathbb{R}^+$ define μ^{ε}, $\varepsilon>0$, to be the geometric product measure with

$$\mu^{\varepsilon}[\eta(x)=k]=\nu_{\rho_0(\varepsilon x)}[\eta(x)=k] \tag{2.4}$$

In Ref. 3 it is proven that $\{\mu^{\varepsilon}\}_{\varepsilon>0}$ defines a local equilibrium distribution with $t(\varepsilon,\tau)=\varepsilon^{-1}\tau$ and equilibrium profile $\rho(\xi,t)$, the solution of the differential equation

$$\partial\rho/\partial t=-(1+\rho)^{-2}\,\partial\rho/\partial\xi \tag{2.5a}$$

with initial condition

$$\rho(\xi,0)=\rho_0(\xi) \tag{2.5b}$$

(2.5a,b) were established for two classes of initial profiles: (i) ρ_0 smooth and decreasing; (ii) ρ_0 increasing and piecewise constant. [In the latter case

(2.5a,b) were shown to hold in the weak sense, and (1.5) to hold at points where $\rho(\cdot, t)$ is continuous.] A particularly simple case in class (ii) is

$$\rho_0(\xi) = \begin{cases} \rho_0, & \xi > 0 \\ 0, & \xi < 0 \end{cases} \tag{2.6}$$

In this case the differential equation predicts that $\rho(\cdot, t)$ is given by

$$\rho(\xi, t) = \rho_0(\xi - v(\rho_0)t) \tag{2.7a}$$

with $v(\rho_0)$ given by

$$v(\rho_0) = (1 + \rho_0)^{-1} \tag{2.7b}$$

This case might be considered to model the propagation of a shock wave.

In Ref. 3 it was conjectured that the state at the shock might not be an extremal equilibrium state. We prove in the next section that the state at the shock is in fact a mixture of form: $\lambda(x, t)\, v_0 + [1 - \lambda(x, t)]\, v_{\rho_0}$. We find the equation satisfied by $\lambda(x, t)$. In the terminology of Ref. 1, the product measure with equilibrium profile given in (2.6) is weakly locally ergodic but not locally ergodic. In the final section we discuss the general case with shocks.

3. THEOREM

Theorem 1. Let μ be the geometric product measure with

$$\mu[\eta(x) = k] = \begin{cases} v_{\rho_0}[\eta(x) = k], & x \geqslant 0 \\ v_0[\eta(x) = k], & x < 0 \end{cases} \tag{3.1}$$

Given $\delta > 0$ there exists $T_\delta < \infty$ such that

$$\lim_{t \to \infty} \sup_x \left\{ \left| P^\mu \left[\bar{d} \left(D_x \frac{1}{T_\delta} \int_t^{t + T_\delta} ds\, \delta_{\eta_s}, v_{\rho_0} \right) > \delta \right] - (1 - \bar{\lambda}(x, t)) \right| \right.$$
$$\left. + \left| P^\mu \left[\bar{d} \left(D_x \frac{1}{T_\delta} \int_t^{t + T_\delta} ds\, \delta_{\eta_s}, v_0 \right) > \delta \right] - \bar{\lambda}(x, t) \right| \right\} = 0 \tag{3.2}$$

$\bar{\lambda}(\xi, t)$ is the solution of the equation

$$\partial\bar{\lambda}/\partial t = -v(\rho_0)\, \partial\bar{\lambda}/\partial\xi + [D(\rho_0)/2]\, \partial^2/\partial\xi^2\, \bar{\lambda} \tag{3.3a}$$

with initial condition

$$\bar{\lambda}(\xi, 0) = \begin{cases} 1, & \xi < 0 \\ 0, & \xi \geqslant 0 \end{cases} \tag{3.3b}$$

and

$$v(\rho_0) = D(\rho_0) = (1 + \rho_0)^{-1} \tag{3.3c}$$

Corollary. For any function T_ε, $T_\varepsilon \to \infty$ as $\varepsilon \to 0$, and any $t > 0$, $\xi \in \mathbb{R}$

$$\lim_{\varepsilon \to 0} \frac{1}{T_\varepsilon} \int_{t\varepsilon^{-1}}^{t\varepsilon^{-1} + T_\varepsilon} ds\, T_s^* D_{v(\rho_0)t\varepsilon^{-1} + \xi\varepsilon^{-1/2}} \mu$$

$$= [1 - \lambda(\xi, t)]\, \nu_{\rho_0} + \lambda(\xi, t)\, \nu_0 \tag{3.4}$$

$\lambda(\xi, t)$ satisfies the equation

$$\partial\lambda/\partial t = [D(\rho_0)/2]\, \partial^2/\partial\xi^2 \lambda \tag{3.5a}$$

with initial condition

$$\lambda(\xi, 0) = \begin{cases} 1, & \xi < 0 \\ 0, & \xi \geqslant 0 \end{cases} \tag{3.5b}$$

Remarks. (1) The corollary follows by averaging over the paths of the process and making use of the equality

$$\bar{\lambda}(\varepsilon^{-1} v(\rho_0)t + \varepsilon^{-1/2}\xi, \varepsilon^{-1}t) = \lambda(\xi, t) \tag{3.6}$$

(2) We can require T_ε to tend to infinity as slowly as we please; in particular, taking $\varepsilon T_\varepsilon \to 0$ we can consider the time averaging in (3.4) to be over a "microscopic" time interval.

(3) The interpretation of the corollary to theorem one is the following. Viewed on a scale of order $\varepsilon^{-1/2}$, intermediate between the microscopic scale (unit length = lattice spacing) and the macroscopic scale (unit length $= \varepsilon^{-1}$), the system is in a statistical mixture of pure states with parameter varying smoothly (on this scale) from zero to one.

Proof of Theorem 1. We will prove the following: For every $A \subset \mathbb{Z}$, $0 < |A| < \infty$ and $\delta > 0$, there exists $T_\delta < \infty$ such that

$$\lim_{t \to \infty} \sup_x \left\{ \left| P^\mu \left[\left| \frac{1}{T_\delta} \int_t^{t+T_\delta} ds\, \eta_s(A + x) - \nu_{\rho_0}(\eta(A)) \right| > \delta \right] \right.\right.$$

$$\left. - [1 - \bar{\lambda}(x, t)] \right| + \left| P^\mu \left[\frac{1}{T_\delta} \int_t^{t+T_\delta} ds\, \eta_s(A + x) < \delta \right] \right.$$

$$\left.\left. - \bar{\lambda}(x, t) \right| \right\} = 0 \tag{3.7}$$

(3.2) follows easily from (3.7).

To begin, consider the motion of a test particle added to the AZRP at $x=0$ at time $t=0$. To simplify the discussion assume that there are no particles to the left of zero. Let z_t be the location of the test particle at time t. We assume that $(\eta_0(x))_{x\geqslant 0}$ is in equilibrium with mean density ρ_0 at time zero. The process (z_t, η_t) is then defined by the following:

(i) η_t, $t\geqslant 0$, is an AZRP.

(ii) Conditional on $\eta_t(z_t)=0$, z_t performs an asymmetric random walk to the right at speed one.

Then we have the following (perhaps surprising) fact:

Proposition 1. Let T_x be the time spent by the test particle at site $x\geqslant 0$: $T_x=|\{t: z_t=x\}|$. Then the T_x, $x\geqslant 0$, are independent and identically distributed with an exponential distribution of mean $(1+\rho_0)$.

Corollary. $\varepsilon^{1/2}(z_{t/\varepsilon}-v(\rho_0)t/\varepsilon] \to D(\rho_0)^{1/2} B_t$ weakly, where B_t, $t\geqslant 0$, is Brownian motion starting at 0, and $D(\rho_0)=v(\rho_0)=(1+\rho_0)^{-1}$.

Proposition 1 follows from a well-known theorem in the statistical theory of queues, originally due to Reich.[7] This theorem states that in a sequence of simple queues in tandem, at equilibrium, the waiting times of a "typical customer" in the queues are independent. The interpretation of "typical customer" may be taken to be one who arrives at the first queue at time zero. This customer finds the queues in equilibrium, so the proposition follows from Reich's theorem.

In Ref. 8 the reader will find a proof of Reich's theorem by a rather clever "time reversal" argument. If this is not sufficiently convincing, the reader should turn to the Appendix of this paper, in which another proof, requiring only a few calculations with the generator, is given.

Turning to the proof of (3.7), let z_t be the location of the "last" η particle, i.e., $z_t=\inf\{z: \eta_t(z)>0\}$. Since the distribution of $\eta_0(x_0)$, conditional on $\eta_0(x_0)>0$, is the same as that of $\eta_0(x_0)+1$, we may represent the process, conditional on $z_0=x_0$, as (η_t, z_t) with η_t in equilibrium and law that described before Proposition 1.

Let $\delta>0$ and choose $T_\delta<\infty$ such that

$$\sup_{x,t} P^{\nu_{\rho_0}}\left[\left|\frac{1}{T_\delta}\int_t^{t+T_\delta} ds\, \eta_s(A+x)-\nu_{\rho_0}(\eta(A))\right|>\delta\right]<\delta \tag{3.8}$$

That T_δ exists follows from the translation invariance, stationarity, and ergodicity of $P^{\nu_{\rho_0}}$ and from von Neumann's L_2-ergodic theorem.

Define, for each $L>0$, $x\geqslant 0$, $t\geqslant 0$, $\delta>0$,

$$\mathscr{E}_{x,t}^{\pm L} := \{\omega\in\Omega: z_s(\omega) \gtrless v(\rho_0)t+(x\pm L)\, t^{1/2},\ t\leqslant s\leqslant t+T_\delta\} \tag{3.9}$$

Let $A \subset \mathbb{Z}$, $0 < |A| < \infty$. On $\mathscr{E}_{x,t}^{+L}$ we have, for t sufficiently large,

$$\frac{1}{T_\delta} \int_t^{t+T_\delta} ds\, \eta_s(A + v(\rho_0)t + xt^{1/2}) = 0 \tag{3.10}$$

and similarily on $\mathscr{E}_{x,t}^{-L}$,

$$\left| \frac{1}{T_\delta} \int_t^{t+T_\delta} ds\, \eta_s(A + v(\rho_0)t + xt^{1/2}) - v_{\rho_0}(\eta(A)) \right| < \delta \tag{3.11}$$

except for a set of paths of total measure less than δ, by (3.8). (To see this, note that we could have added η' particles to the left of zero at time zero and coupled the motion so that (i) $\eta + \eta'$ is in equilibrium (v_{ρ_0}), (ii) η' particles never bypass η particles (see Ref. 3). On $\mathscr{E}_{x,t}^{-L}$ we could replace η_t by $\eta_t + \eta_t'$.)

Finally, we use Proposition 1, the definition of weak convergence, and the continuity of the sample paths of Brownian motion to show that ($t = \varepsilon^{-1}$):

$$\begin{aligned} &P^\mu\{\omega \in \Omega : \sqrt{\varepsilon}\, [z_{s\varepsilon^{-1}} - v(\rho_0)\, s\varepsilon^{-1}] \gtrless x \pm L + O(\varepsilon) \\ &\qquad 1 \leqslant s \leqslant 1 + O(\varepsilon)\} \\ &\quad = P^\mu[\mathscr{E}_{x,t}^{\pm L}] \xrightarrow[t \to \infty]{} P[D(\rho_0)^{1/2}\, B_1 \gtrless x \pm L] \end{aligned} \tag{3.12}$$

Substituting $v(\rho_0)t + xt^{1/2}$ for x in (3.7), using the identity

$$\bar{\lambda}(v(\rho_0)t + xt^{1/2}, t) = P[D(\rho_0)^{1/2}\, B_1 > x] \tag{3.13}$$

and taking the limits: $t \to \infty$, then $L \to 0$ completes the proof of the theorem. ∎

4. REMARKS.—THE GENERAL INCREASING INITIAL EQUILIBRIUM PROFILE

The methods of Ref. 3 suffice also to treat the case of an arbitrary increasing initial equilibrium profile. In this section a theorem to this effect is stated and its proof is sketched.

Theorem 2. Let μ^ε be given by (2.4) with ρ_0 increasing and piecewise continuous. Let $\rho(\xi, t)$ be the weak solution of (2.5a,b) "with entropy condition."[4] Then (1.5) holds for all (x, t) at which $\rho(\cdot, t)$ is continuous.

Sketch of Proof. The case of ρ_0 piecewise *constant* was treated in Ref. 3. Approximate $\rho_0(\cdot)$ above and below by piecewise-constant $\rho_{0,\delta}^{\pm}$ (with mesh of size δ). Use the following facts:

(i) Let $\rho_i(\cdot)$, $i=1, 2$ be increasing with $\rho_1(\cdot) \leqslant \rho_2(\cdot)$. If $\rho_1(\cdot, t)$ and $\rho_2(\cdot, t)$ are the weak solutions of (2.5a) with initial conditions $\rho_1(\cdot)$ and $\rho_2(\cdot)$, respectively, satisfying the entropy condition,[4] then $\rho_1(\xi, t) \leqslant \rho_2(\xi, t)$ at all points of continuity of $\rho_1(\cdot, t)$ and $\rho_2(\cdot, t)$ and all $t > 0$.

(ii) The solution of (2.5) is stable in L_1 norm.

(iii) Let $\mu_1 \leqslant \mu_2$ in the FKG (stochastic) sense. Then $T_t^* \mu_1 \leqslant T_t^* \mu_2$ for all $t \geqslant 0$ if T_t^* denotes time evolution of measures in the AZRP.

The proof of (i) is immediate from the geometric solution (see Ref. 4). (ii) is proved in Ref. 9. The proof of (iii) is just the existence of the coupling used in Ref. 3.

To complete the proof of Theorem 2 use the fact that $\rho_1(\cdot) < \rho_2(\cdot)$ implies the corresponding inequality for states, (i)–(iii), and the results in Ref. 3 to conclude that (1.5) holds except for the countable set of points at which the approximate solutions of (2.5) have discontinuities. From the geometric solution of (2.5) it is clear that this set is, for each time t and $\delta > 0$, a continuous function of the set of initial jumps, so that varying this set (for each δ) we conclude that (1.5) holds everywhere except at the jumps of the actual solution [with initial condition $\rho_0(\cdot)$]. ∎

5. CONCLUSION

The interest in this result is that an initially continuous (even C^∞) increasing equilibrium profile may develop a shock at a later time. In fact it is easy to construct a continuous $\rho_0(\cdot)$ such that $\rho_0(\cdot, t)$ is the traveling shock solution (2.7) after a time $T_{(\text{shock})}$. It is natural to conjecture that the state at the shock—once it forms—is a statistical mixture as described in Theorem 1 and its corollary.

The truth of this conjecture would indeed establish the appropriateness of the term "dynamical phase transition" in this context. Unfortunately the methods used to prove Theorem 1 do not generalize in any straightforward way to even a two-step shock, let alone to the fully nonequilibrium situation we encounter when trying to deal with the general increasing case.

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APPENDIX. A PROOF OF REICH'S THEOREM

We provide a new proof of a geometrical character, starting directly with the generator. The idea of the proof is to keep the "customers" fixed and let the "servers" jump past the customers in the reverse direction. If we can show that the process formed by servers jumping past a fixed customer is Poisson we shall have proved Reich's theorem.

Lemma. Let $S_t = (S_t(x))_{x \in \mathbb{Z}+1/2}$ be an AZRP on $\mathbb{Z}+1/2$, jumping to the left, in equilibrium with mean density ρ_0^{-1}. Let $y_t(n) \in \mathbb{Z}+1/2$ be the location of the nth S particle to the right of zero at time t. [That is, $y_t(n+1) \geqslant y_t(n) \geqslant 0$ and $S_t(x) = |\{n: y_t(n) = x\}|$.] Let (η_t, z_t) be the coupled process described before Proposition 1. Then

$$(\eta_t(z_t), \eta_t(z_t+n))_{n \geqslant 1} \sim (y_t(1) - 1/2, y_t(n+1) - y_t(n))_{n \geqslant 1} \tag{A1}$$

i.e., the joint distributions are identical.

Proof of the Lemma. Let $\zeta_t(x) = \eta_t(z_t + x)$. $\zeta_t(x)$ represents the process as seen from the moving frame of the particle. A short calculation using (2.1) shows that $\zeta_t(x)$, $x \geqslant 0$ is Markovian with generator (let f be a cylinder function):

$$\begin{aligned}[\hat{L}f](\zeta) = \sum_{x \geqslant 0} & 1_{[\zeta(x)>0]}[f(\zeta^{x,x+1}) - f(\zeta)] \\ & + 1_{[\zeta(0)=0]}[D_{-1}f(\zeta) - f(\zeta)]\end{aligned} \tag{A2}$$

The generator of the process $(y_t(1) - 1/2, y_t(2) - y_t(1),...)$ is easily seen to be equal to $\hat{L}$ as well.

Thus to complete the proof of the lemma one only needs to check the equality of the distributions at time zero. We have $(\alpha = \rho_0(1+\rho_0)^{-1})$:

$$P[\eta_0(x) = k] = \alpha^k(1-\alpha)$$

$$P[S_0(x+1/2) = k] = (1-\alpha)^k \alpha$$

$$\begin{aligned} P[y_0(1) - 1/2 = k_1, & \; y_0(2) - y_0(1) = k_2,..., y_0(n) - y_0(n-1) = k_n] \\ &= [\alpha^{k_1}(1-\alpha)\alpha][\alpha^{k_2}(1-\alpha)] \cdots [\alpha^{k_{n-1}}(1-\alpha)][\alpha^{k_n-1}(1-\alpha)] \\ &= (1-\alpha)^n \alpha^{\sum_{i=1}^n k_i} \\ &= P[\eta_0(0) = k_1,..., \eta_0(n-1) = k_n] \end{aligned} \tag{A3}$$

This proves the lemma. ∎

To complete the proof of Reich's theorem note that a jump: $z_t \to z_t + 1$ is signaled by a shift in the configuration $\zeta_t(x)$, $x \geqslant 0$ to the left. In the realization of ζ_t in terms of the S_t process, this shift occurs precisely when a server (S particle) jumps past zero. That this process is Poisson (a theorem of Burke; see Ref. 8) is established by first showing that a single queue (site) in the AZRP, in equilibrium, is simply a reversible, birth-and-death process. Hence the "leaving" process is Poisson.

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